

# SPATIAL SORTING

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## Online Appendix

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#### I. General Technology: $N$ skill types

The firm's problem with  $N$  skills is given by:

$$\pi(m_{1j}, \dots, m_{jN}) = A_j F(m_{1j}, \dots, m_{jN}) - \sum_{i=1}^N w_{ij} m_{ij} \quad (\text{A.1})$$

Then, the system becomes:

$$\left\{ \begin{array}{l} A_1 F_i(m_{11}, \dots, m_{N1}) = \left(\frac{p_1}{p_2}\right)^\alpha A_2 F_i(m_{12}, \dots, m_{N2}), \quad \forall i \in \{1, \dots, N\} \\ \sum_{i=1}^N F_i(m_{11}, \dots, m_{N1}) m_{i1} = H \frac{p_1}{\alpha A_1} \\ \sum_{i=1}^N F_i(m_{12}, \dots, m_{N2}) m_{i2} = H \frac{p_2}{\alpha A_2} \\ C_1 m_{i1} + C_2 m_{i2} = M_i, \quad \forall i \in \{1, \dots, N\} \end{array} \right. \quad (\text{A.2})$$

Now, define  $F(\cdot)$  as (assuming without loss that  $N$  is even):

$$F(\cdot) = (m_{11}^\gamma y_1 + m_{N1}^\gamma y_N)^{\lambda_1} + (m_{21}^\gamma y_2 + m_{N-1,1}^\gamma y_{N-1})^{\lambda_2} + \dots + (m_{\frac{N}{2},1}^\gamma y_{\frac{N}{2}} + m_{\frac{N}{2}+1,1}^\gamma y_{\frac{N}{2}+1})^{\lambda_{\frac{N}{2}}} \quad (\text{A.3})$$

Substituting this back into the system, we have:

$$\left\{ \begin{array}{l} \frac{A_1 (m_{i,1}^\gamma y_i + m_{N-(i-1),1}^\gamma y_{N-(i-1)})^{\lambda_{\min\{i, N-(i-1)\}} m_{i,1}^{\gamma-1}}}{A_2 (m_{i,2}^\gamma y_i + m_{N-(i-1),2}^\gamma y_{N-(i-1)})^{\lambda_{\min\{i, N-(i-1)\}} m_{i,2}^{\gamma-1}}} = \left(\frac{p_1}{p_2}\right)^\alpha, \quad \forall i \in \{1, \dots, N\} \quad (1), \dots, (N) \\ \sum_{i=1}^N \lambda_{\min\{i, N-(i-1)\}} (m_{i,1}^\gamma y_i + m_{N-(i-1),1}^\gamma y_{N-(i-1)})^{\lambda_{\min\{i, N-(i-1)\}} m_{i,1}^\gamma y_i = H \frac{p_1}{\alpha \beta^\gamma A_1} \quad (N+1) \\ \sum_{i=1}^N \lambda_{\min\{i, N-(i-1)\}} (m_{i,2}^\gamma y_i + m_{N-(i-1),2}^\gamma y_{N-(i-1)})^{\lambda_{\min\{i, N-(i-1)\}} m_{i,2}^\gamma y_i = H \frac{p_2}{\alpha \beta^\gamma A_2} \quad (N+2) \\ C_1 m_{i1} + C_2 m_{i2} = M_i, \quad \forall i \in \{1, \dots, N\} \quad (N+3), \dots, (2N+2) \end{array} \right. \quad (\text{A.4})$$

From the first  $N$  equations, dividing the expressions for  $i$  and  $N - (i - 1)$ , we have:

$$m_{i,1} = \frac{M_i}{M_{N-(i-1)}} m_{N-(i-1),1}, \text{ for } i \in \left\{1, \dots, \frac{N}{2}\right\} \quad (\text{A.5})$$

Considering a symmetric distribution (so  $M_i = M_{N-(i-1)}$ , for  $i \in \{1, \dots, \frac{N}{2}\}$ ), we have that  $m_{i,1} = m_{N-(i-1),1}$ . Similarly  $m_{i,2} = m_{N-(i-1),2} = \frac{M_i}{N_2} - \frac{N_1}{N_2} m_{i,1}$ , for  $i \in \{1, \dots, \frac{N}{2}\}$ .

From equations for  $i$  and  $j$  for  $i \neq j$  from the first  $N$  equations, we have:

$$\left[ \frac{m_{i,1}}{m_{N-(i-1),1}} \right]^{\gamma-1} = \left[ \frac{m_{i,2}}{m_{N-(i-1),2}} \right]^{\gamma-1}, \text{ for } i \in \left\{ 1, \dots, \frac{N}{2} \right\} \quad (\text{A.6})$$

Using equations  $(N+3)$  to  $(2N+2)$ , we have:

$$m_{i,1} = \frac{M_i}{M_{N-(i-1)}} m_{N-(i-1),1}, \text{ for } i \in \left\{ 1, \dots, \frac{N}{2} \right\}$$

Considering a symmetric distribution (so  $M_i = M_{N-(i-1)}$ , for  $i \in \{1, \dots, \frac{N}{2}\}$ ), we have that  $m_{i,1} = m_{N-(i-1),1}$ . Similarly  $m_{i,2} = m_{N-(i-1),2} = \frac{M_i}{C_2} - \frac{C_1}{C_2} m_{i,1}$ , for  $i \in \{1, \dots, \frac{N}{2}\}$ .

From the first  $N$  equations, we have:

$$m_{i,1} = \frac{\left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \right]^{\frac{1}{\lambda_{\min\{i, N-(i-1)\}_i} \gamma^{-1}}} M_i}{C_2 + \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \right]^{\frac{1}{\lambda_{\min\{i, N-(i-1)\}_i} \gamma^{-1}}} C_1} \quad (\text{A.7})$$

Similarly:

$$m_{i,2} = \frac{M_i}{C_2 + \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \right]^{\frac{1}{\lambda_{\min\{i, N-(i-1)\}_i} \gamma^{-1}}} C_1} \quad (\text{A.8})$$

Then, from  $(N+1)$ , we have:

$$\frac{A_1}{p_1} \sum_{i=1}^N \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \right]^{\frac{\lambda_{\min\{i, N-(i-1)\}_i} \gamma}{\lambda_{\min\{i, N-(i-1)\}_i} \gamma^{-1}}} \left( \frac{M_i}{C_2 + \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \right]^{\frac{1}{\lambda_{\min\{i, N-(i-1)\}_i} \gamma^{-1}}} C_1} \right)^{\lambda_{\min\{i, N-(i-1)\}_i} \gamma} \right. \\ \left. \times \lambda_{\min\{i, N-(i-1)\}_i} (y_i + y_{N-(i-1)})^{\lambda_{\min\{i, N-(i-1)\}_i} - 1} y_i \right\} = \frac{H}{\alpha \beta \gamma} \quad (\text{A.9})$$

Similarly, from  $(N+2)$ , we have:

$$\frac{A_2}{p_2} \sum_{i=1}^N \left\{ \left( \frac{M_i}{C_2 + \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \right]^{\frac{1}{\lambda_{\min\{i, N-(i-1)\}_i} \gamma^{-1}}} C_1} \right)^{\lambda_{\min\{i, N-(i-1)\}_i} \gamma} \right. \\ \left. \times \lambda_{\min\{i, N-(i-1)\}_i} (y_i + y_{N-(i-1)})^{\lambda_{\min\{i, N-(i-1)\}_i} - 1} y_i \right\} = \frac{H}{\alpha \beta \gamma} \quad (\text{A.10})$$

Combining these expressions and rearranging, we have:

$$\sum_{i=1}^N \left\{ \left[ \left( \frac{p_2}{p_1} \right)^{\frac{1-(1-\alpha)\lambda_{\min\{i, N-(i-1)\}_i} \gamma}{1-\lambda_{\min\{i, N-(i-1)\}_i} \gamma}} \left( \frac{A_1}{A_2} \right)^{\frac{1}{1-\lambda_{\min\{i, N-(i-1)\}_i} \gamma}} - 1 \right] \times \right. \\ \left. \times \left( \frac{M_i}{C_2 + \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \right]^{\frac{1}{\lambda_{\min\{i, N-(i-1)\}_i} \gamma^{-1}}} C_1} \right)^{\lambda_{\min\{i, N-(i-1)\}_i} \gamma} \right. \\ \left. \times \lambda_{\min\{i, N-(i-1)\}_i} (y_i + y_{N-(i-1)})^{\lambda_{\min\{i, N-(i-1)\}_i} - 1} y_i \right\} = 0 \quad (\star) \quad (\text{A.11})$$

**Lemma A.1:** Let  $\lambda_i > 1$ ,  $\lambda_i \gamma < 1$ , for every  $i \in \{1, 2, \dots, \frac{N}{2}\}$  and  $\{\lambda_i\}_{i=\frac{N}{2}}$  be a decreasing sequence. If  $A_1 > A_2$ , then house prices are higher in the city with higher TFP,  $p_1 > p_2$ .

**Proof:** In order to satisfy the equality (★), the only terms that can be negative are the ones in between squared brackets. Since  $\frac{A_1}{A_2} > 1$  and  $\min_i \left\{ \frac{1}{1 - \lambda_{\min\{i, N-(i-1)\}} \gamma} \right\} > 1$ , the only way one of these terms is negative is if:

$$\min_i \left\{ \left( \frac{p_2}{p_1} \right)^{\frac{1 - (1-\alpha)\lambda_{\min\{i, N-(i-1)\}} \gamma}{1 - \lambda_{\min\{i, N-(i-1)\}} \gamma}} \left( \frac{A_1}{A_2} \right)^{\frac{1}{1 - \lambda_{\min\{i, N-(i-1)\}} \gamma}} \right\} < 1 \quad (\text{A.12})$$

But, since  $\min_i \left\{ \left( \frac{p_2}{p_1} \right)^{\frac{1 - (1-\alpha)\lambda_{\min\{i, N-(i-1)\}} \gamma}{1 - \lambda_{\min\{i, N-(i-1)\}} \gamma}} \left( \frac{A_1}{A_2} \right)^{\frac{1}{1 - \lambda_{\min\{i, N-(i-1)\}} \gamma}} \right\} > 1$ , the only way that the above inequality is satisfied is if  $\frac{p_2}{p_1} < 1 \Rightarrow p_2 < p_1$ . ■

**Theorem A.1:** City Size and TFP. Let  $A_1 > A_2$ ,  $\lambda_i > 1$ ,  $\lambda_i \gamma < 1$ , for every  $i \in \{1, 2, \dots, \frac{N}{2}\}$  and  $\{\lambda_i\}_{i=1}^{\frac{N}{2}}$  be a decreasing sequence. Then, the more productive city is larger,  $S_1 > S_2$ .

**Proof:** Based on Lemma A.1, we know that  $p_1 > p_2$ . Since  $\lambda_i > 1$ ,  $\lambda_i \gamma < 1$ , for every  $i \in \{1, 2, \dots, \frac{N}{2}\}$  and  $\{\lambda_i\}_{i=1}^{\frac{N}{2}}$  is a decreasing sequence, we know that  $\frac{1}{1 - \lambda_1 \gamma} > \frac{1}{1 - \lambda_2 \gamma} > \dots > \frac{1}{1 - \lambda_{\frac{N}{2}} \gamma} > 1$  and  $\frac{1 - (1-\alpha)\lambda_1 \gamma}{1 - \lambda_1 \gamma} > \frac{1 - (1-\alpha)\lambda_{\frac{N}{2}} \gamma}{1 - \lambda_{\frac{N}{2}} \gamma} > 1$ . But then, in order to satisfy (★), we must have some positive and negative terms. The term with respect to  $i = \frac{N}{2}$  is positive if:

$$\frac{p_2}{p_1} > \left( \frac{A_2}{A_1} \right)^{\frac{1}{1 - (1-\alpha)\lambda_{\frac{N}{2}} \gamma}} \quad (\text{A.13})$$

While the term with respect to  $i = 1$  is positive if:

$$\frac{p_2}{p_1} > \left( \frac{A_2}{A_1} \right)^{\frac{1}{1 - (1-\alpha)\lambda_1 \gamma}} \quad (\text{A.14})$$

But notice that  $\frac{A_2}{A_1} < 1$ . Then, we have that:

$$\left( \frac{A_2}{A_1} \right)^{\frac{1}{1 - (1-\alpha)\lambda_{\frac{N}{2}} \gamma}} > \left( \frac{A_2}{A_1} \right)^{\frac{1}{1 - (1-\alpha)\lambda_1 \gamma}} \quad (\text{A.15})$$

Therefore, in order to satisfy (★), we must have that:

$$\frac{p_2}{p_1} \in \left( \left( \frac{A_2}{A_1} \right)^{\frac{1}{1 - (1-\alpha)\lambda_1 \gamma}}, \left( \frac{A_2}{A_1} \right)^{\frac{1}{1 - (1-\alpha)\lambda_{\frac{N}{2}} \gamma}} \right) \quad (\text{A.16})$$

But this implies that:

$$\left( \frac{p_2}{p_1} \right)^\alpha \frac{A_1}{A_2} \in \left( \left( \frac{A_2}{A_1} \right)^{\frac{\alpha}{1 - (1-\alpha)\lambda_1 \gamma} - 1}, \left( \frac{A_2}{A_1} \right)^{\frac{\alpha}{1 - (1-\alpha)\lambda_{\frac{N}{2}} \gamma} - 1} \right) \quad (\text{A.17})$$

Rearranging it, we have:

$$\left( \frac{p_2}{p_1} \right)^\alpha \frac{A_1}{A_2} \in \left( \left( \frac{A_1}{A_2} \right)^{\frac{(1-\alpha)(1-\lambda_1 \gamma)}{1 - (1-\alpha)\lambda_1 \gamma}}, \left( \frac{A_1}{A_2} \right)^{\frac{(1-\alpha)(1-\lambda_{\frac{N}{2}} \gamma)}{1 - (1-\alpha)\lambda_{\frac{N}{2}} \gamma}} \right) \quad (\text{A.18})$$

Since  $A_1 > A_2$ , we have that:

$$\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2} > 1 \quad (\text{A.19})$$

From the expressions for  $m_{ij}$ :

$$m_{i,1} = \frac{\left[\left(\frac{p_1}{p_2}\right)^\alpha \frac{A_2}{A_1}\right]^{\frac{1}{\lambda_{\min\{i, N-(i-1)\}}_i \gamma - 1}} M_i}{C_2 + \left[\left(\frac{p_1}{p_2}\right)^\alpha \frac{A_2}{A_1}\right]^{\frac{1}{\lambda_{\min\{i, N-(i-1)\}}_i \gamma - 1}} C_1} \quad (\text{A.20})$$

$$m_{i,2} = \frac{M_i}{C_2 + \left[\left(\frac{p_1}{p_2}\right)^\alpha \frac{A_2}{A_1}\right]^{\frac{1}{\lambda_{\min\{i, N-(i-1)\}}_i \gamma - 1}} C_1} \quad (\text{A.21})$$

and  $\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2} > 1$ , we have that  $m_{i,1} > m_{i,2}$ , for every  $i \in \{1, 2, \dots, N\}$ . Finally, since:

$$S_j = \sum_{i=1}^N m_{ij} \quad (\text{A.22})$$

it immediately follows that  $S_1 > S_2$ . ■

**Theorem A.2:** thick tails. Given that  $A_1 > A_2$ ,  $\lambda_i > 1$ ,  $\lambda_i \gamma < 1$ , for every  $i \in \{1, 2, \dots, \frac{N}{2}\}$  and  $\{\lambda_i\}_{i=1}^{\frac{N}{2}}$  is a decreasing sequence, the skill distribution in the larger city has thicker tails.

**Proof:**

$$pdf_{11} = \frac{\left[\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2}\right]^{\frac{1}{1-\lambda_1 \gamma}} M_1}{C_2 + \left[\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2}\right]^{\frac{1}{1-\lambda_1 \gamma}} C_1} \quad (\text{A.23})$$

$$\sum_{i=1}^N \frac{\left[\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2}\right]^{\frac{1}{1-\lambda_{\min\{i, N-(i-1)\}}_i \gamma}} M_i}{C_2 + \left[\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2}\right]^{\frac{1}{1-\lambda_{\min\{i, N-(i-1)\}}_i \gamma}} C_1}$$

$$= \frac{1}{\sum_{i=1}^N \frac{M_i}{M_1} \left\{ \left[\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2}\right]^{\frac{(\lambda_{\min\{i, N-(i-1)\}}_i - \lambda_1) \gamma}{(1-\lambda_{\min\{i, N-(i-1)\}}_i \gamma)(1-\lambda_1 \gamma)}} \times \right.} \quad (\text{A.24})$$

$$\left. \times \frac{C_2 + \left[\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2}\right]^{\frac{1}{1-\lambda_1 \gamma}} C_1}{C_2 + \left[\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2}\right]^{\frac{1}{1-\lambda_{\min\{i, N-(i-1)\}}_i \gamma}} C_1} \right\}}$$

while:

$$pdf_{12} = \frac{\frac{M_1}{C_2 + \left[\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2}\right]^{\frac{1}{1-\lambda_1 \gamma}} C_1}}{\sum_{i=1}^N \frac{M_i}{C_2 + \left[\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2}\right]^{\frac{1}{1-\lambda_{\min\{i, N-(i-1)\}}_i \gamma}} C_1}} \quad (\text{A.25})$$

$$= \frac{1}{\sum_{i=1}^N \frac{M_i}{M_1} \frac{C_2 + \left[\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2}\right]^{\frac{1}{1-\lambda_1 \gamma}} C_1}{C_2 + \left[\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2}\right]^{\frac{1}{1-\lambda_{\min\{i, N-(i-1)\}}_i \gamma}} C_1}} \quad (\text{A.26})$$

But then, since  $\lambda_1 = \max\{\lambda_i\}_{i=1}^N$  and  $\left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2} > 1$ , we have that:

$$\sum_{i=1}^N \frac{M_i}{M_1} \left\{ \begin{aligned} & \left[ \left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2} \right]^{\frac{(\lambda_{\min\{i, N-(i-1)\}_i - \lambda_1)\gamma}{(1-\lambda_{\min\{i, N-(i-1)\}_i)\gamma} (1-\lambda_1\gamma)}} \times \\ & \times \frac{C_2 + \left[ \left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2} \right]^{\frac{1}{1-\lambda_1\gamma}} C_1}{C_2 + \left[ \left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2} \right]^{\frac{1}{1-\lambda_{\min\{i, N-(i-1)\}_i}\gamma}} C_1} \end{aligned} \right\} \quad (\text{A.27})$$

$$< \sum_{i=1}^N \frac{M_i}{M_1} \frac{C_2 + \left[ \left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2} \right]^{\frac{1}{1-\lambda_1\gamma}} C_1}{C_2 + \left[ \left(\frac{p_2}{p_1}\right)^\alpha \frac{A_1}{A_2} \right]^{\frac{1}{1-\lambda_{\min\{i, N-(i-1)\}_i}\gamma}} C_1} \quad (\text{A.28})$$

Therefore,  $pdf_{11} > pdf_{12}$ . Since the distributions are symmetric, we also have  $pdf_{N1} > pdf_{N2}$ . ■

## II. Nested CES and Free Entry of firms

We now consider a technology with gross complementarities  $\beta$  and 3 skill types:

$$Y = A_1 \left[ m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda \right]^\beta. \quad (\text{A.29})$$

In this model we simultaneously consider the additional extension that firms are perfectly mobile. Firms can relocate instantaneously and at no cost to another city. To establish itself in a city, a firm must buy a amount  $k$  of land. Given that firms can freely enter and exit cities, we have that in equilibrium, firms must generate zero profits, i.e.:

$$A_j F(m_{1j}, m_{2j}, m_{3j}) - \sum_i^3 w_{ij} m_{ij} - k p_j = 0, \forall j \in \{1, 2\} \quad (\text{A.30})$$

We will assume that there are only two cities, 1 and 2, while city  $i$  has a measure  $N_i$  of firms, that will be pin down in equilibrium. Since  $w_{i2} = \left(\frac{p_2}{p_1}\right)^\alpha w_{i1}$ , the system then becomes:

$$\left\{ \begin{array}{l} A_1 \frac{\left[ \begin{array}{l} [m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda]^{\beta-1} \times \\ \times [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^{\lambda-1} \end{array} \right]}{\left[ \begin{array}{l} [m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda]^{\beta-1} \times \\ \times [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^{\lambda-1} \end{array} \right]} m_{11}^{\gamma-1} = \left(\frac{p_1}{p_2}\right)^\alpha A_2 m_{12}^{\gamma-1} \quad (1) \\ A_1 \frac{\left[ \begin{array}{l} [m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda]^{\beta-1} \\ \times [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^{\lambda-1} \end{array} \right]}{\left[ \begin{array}{l} [m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda]^{\beta-1} \times \\ \times [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^{\lambda-1} \end{array} \right]} m_{21}^{\gamma-1} = \left(\frac{p_1}{p_2}\right)^\alpha A_2 \left[ m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right]^{\beta-1} m_{22}^{\gamma-1} \quad (2) \\ A_1 \frac{\left[ \begin{array}{l} [m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda]^{\beta-1} \times \\ \times [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^{\lambda-1} \end{array} \right]}{\left[ \begin{array}{l} 2 \left[ m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right]^{\beta-1} \times \\ \times [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^{\lambda-1} \end{array} \right]} m_{31}^{\gamma-1} = \left(\frac{p_1}{p_2}\right)^\alpha A_2 m_{32}^{\gamma-1} \quad (3) \\ N_1 m_{i1} + N_2 m_{i2} = M_i, \forall i \in \{1, 2, 3\} \quad (4, 5, 6) \\ \left[ m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda \right]^{\beta-1} \left\{ \lambda [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda + m_{21}^\gamma y_2 \right\} = \left[ \frac{H}{N_1} - k \right] \frac{p_1}{\alpha \gamma \beta A_1} \quad (7) \\ \left[ m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right]^{\beta-1} \left\{ \lambda [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda + m_{22}^\gamma y_2 \right\} = \left[ \frac{H}{N_2} - k \right] \frac{p_2}{\alpha \gamma \beta A_2} \quad (8) \\ \left[ m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda \right]^{\beta-1} \left\{ (1 - \lambda \gamma \beta) [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda + (1 - \gamma \beta) m_{21}^\gamma y_2 \right\} = \frac{k}{A_1} p_1 \quad (9) \\ \left[ m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right]^{\beta-1} \left\{ (1 - \lambda \gamma \beta) [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda + (1 - \gamma \beta) m_{22}^\gamma y_2 \right\} = \frac{k}{A_2} p_2 \quad (10) \end{array} \right. \quad (\text{A.31})$$

From eq. (1) and (3), we have:

$$\begin{aligned} \left[ \frac{m_{11}}{m_{31}} \right]^{\gamma-1} &= \left[ \frac{m_{12}}{m_{32}} \right]^{\gamma-1} \\ \frac{m_{11}}{m_{31}} &= \frac{m_{12}}{m_{32}} \end{aligned}$$

Since:

$$m_{12} = \frac{M_1}{N_2} - \frac{N_1}{N_2} m_{11} \quad (\text{A.32})$$

$$m_{32} = \frac{M_3}{N_2} - \frac{N_1}{N_2} m_{31} \quad (\text{A.33})$$

Substituting it, we have:

$$\frac{m_{11}}{m_{31}} = \frac{M_1 - N_1 m_{11}}{M_3 - N_1 m_{31}} \quad (\text{A.34})$$

Rearranging:

$$m_{11} = \frac{M_1}{M_3} m_{31} \quad (\text{A.35})$$

Considering a symmetric distribution (so  $M_1 = M_3$ ), we have that  $m_{11} = m_{31}$ . Similarly  $m_{12} = m_{32} = \frac{M_1}{N_2} - \frac{N_1}{N_2} m_{11}$ .

From (1) and (2), we have:

$$\frac{[m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^{\lambda-1} m_{11}^{\gamma-1}}{m_{21}^{\gamma-1}} = \frac{[m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^{\lambda-1} m_{12}^{\gamma-1}}{m_{22}^{\gamma-1}} \quad (\text{A.36})$$

Using the symmetry of the distribution and consequentially that  $m_{11} = m_{31}$  and  $m_{12} = m_{32}$ , we have:

$$m_{11}^{\gamma(\lambda-1)} \left( \frac{m_{11}}{m_{21}} \right)^{\gamma-1} = m_{12}^{\gamma(\lambda-1)} \left( \frac{m_{12}}{m_{22}} \right)^{\gamma-1} \quad (\text{A.37})$$

Then:

$$m_{21} = \left( \frac{m_{11}}{m_{12}} \right)^{\frac{\lambda\gamma-1}{\gamma-1}} m_{22} \quad (\text{A.38})$$

Then, from (7) and (9), we have:

$$\frac{\left\{ \lambda [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda + m_{21}^\gamma y_2 \right\}}{\left\{ (1 - \lambda\gamma\beta) [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda + (1 - \gamma\beta) m_{21}^\gamma y_2 \right\}} = \frac{\left[ \frac{H}{N_1} - k \right]}{k\alpha\gamma\beta} \quad (\text{A.39})$$

Using symmetry and again that  $m_{11} = m_{31}$ , we have:

$$m_{21}^\gamma y_2 = \frac{\left\{ (1 - \lambda\gamma\beta) \left[ \frac{H}{N_1 k\alpha\gamma\beta} - \frac{k}{k\alpha\gamma\beta} \right] - \lambda \right\}}{\left\{ 1 - (1 - \gamma\beta) \left[ \frac{H}{N_1 k\alpha\gamma\beta} - \frac{k}{k\alpha\gamma\beta} \right] \right\}} m_{11}^{\lambda\gamma} [y_3 + y_1]^\lambda \quad (\text{A.40})$$

Similarly, from (8) and (10), we have:

$$\frac{\left\{ \lambda [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda + m_{22}^\gamma y_2 \right\}}{\left\{ (1 - \lambda\gamma\beta) [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda + (1 - \gamma\beta) m_{22}^\gamma y_2 \right\}} = \frac{\left[ \frac{H}{N_2} - k \right]}{k\alpha\gamma\beta} \quad (\text{A.41})$$

Using symmetry and again that  $m_{12} = m_{32}$ , we have:

$$m_{22}^\gamma y_2 = \frac{\left\{ (1 - \lambda\gamma\beta) \left[ \frac{H}{N_2 k \alpha \gamma \beta} - \frac{k}{k \alpha \gamma \beta} \right] - \lambda \right\}}{\left\{ 1 - (1 - \gamma\beta) \left[ \frac{H}{N_2 k \alpha \gamma \beta} - \frac{k}{k \alpha \gamma \beta} \right] \right\}} m_{12}^{\lambda\gamma} [y_3 + y_1]^\lambda \quad (\text{A.42})$$

Then, from equaiton (1), we have - again using symmetry:

$$\left[ m_{21}^\gamma y_2 + m_{11}^{\lambda\gamma} [y_3 + y_1]^\lambda \right]^{\beta-1} m_{11}^{\lambda\gamma-1} = \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ m_{22}^\gamma y_2 + m_{12}^{\lambda\gamma} [y_3 + y_1]^\lambda \right]^{\beta-1} m_{12}^{\lambda\gamma-1} \quad (\text{A.43})$$

Substituting  $m_{21}^\gamma y_2$  and  $m_{22}^\gamma y_2$ , we have:

$$\begin{aligned} & \left[ \frac{\beta\gamma(\lambda-1)[(1-\alpha)kN_1-H]}{[1-(1-\alpha)\beta\gamma]kN_1-(1-\beta\gamma)H} \right]^{\beta-1} m_{11}^{\lambda\gamma\beta-1} = \\ & = \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ \frac{\beta\gamma(\lambda-1)[(1-\alpha)kN_2-H]}{[1-(1-\alpha)\beta\gamma]kN_2-(1-\beta\gamma)H} \right]^{\beta-1} m_{12}^{\lambda\gamma\beta-1} \end{aligned}$$

Assuming  $\lambda \neq 1$ , we have:

$$\left( \frac{m_{11}}{m_{12}} \right) = \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ \frac{[1-(1-\alpha)\beta\gamma]kN_1-(1-\beta\gamma)H}{[1-(1-\alpha)\beta\gamma]kN_2-(1-\beta\gamma)H} \times \frac{(1-\alpha)kN_2-H}{(1-\alpha)kN_1-H} \right]^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}} \quad (\text{A.44})$$

Substituting  $m_{12} = \frac{M_1}{N_2} - \frac{N_1}{N_2} m_{11}$ , we have:

$$m_{11} = \frac{\left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ \frac{[1-(1-\alpha)\beta\gamma]kN_1-(1-\beta\gamma)H}{[1-(1-\alpha)\beta\gamma]kN_2-(1-\beta\gamma)H} \times \frac{(1-\alpha)kN_2-H}{(1-\alpha)kN_1-H} \right]^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}}}{\left[ N_2 + N_1 \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ \frac{[1-(1-\alpha)\beta\gamma]kN_1-(1-\beta\gamma)H}{[1-(1-\alpha)\beta\gamma]kN_2-(1-\beta\gamma)H} \times \frac{(1-\alpha)kN_2-H}{(1-\alpha)kN_1-H} \right]^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}} \right]} M_1 \quad (\text{A.45})$$

Since the distribution is symmetric, we have:

$$m_{31} = \frac{\left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ \frac{[1-(1-\alpha)\beta\gamma]kN_1-(1-\beta\gamma)H}{[1-(1-\alpha)\beta\gamma]kN_2-(1-\beta\gamma)H} \times \frac{(1-\alpha)kN_2-H}{(1-\alpha)kN_1-H} \right]^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}}}{\left[ N_2 + N_1 \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ \frac{[1-(1-\alpha)\beta\gamma]kN_1-(1-\beta\gamma)H}{[1-(1-\alpha)\beta\gamma]kN_2-(1-\beta\gamma)H} \times \frac{(1-\alpha)kN_2-H}{(1-\alpha)kN_1-H} \right]^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}} \right]} M_3 \quad (\text{A.46})$$

Finally, from our expression for  $m_{21}^\gamma y_2$ , we have:



$$m_{21}^\gamma y_2 = \frac{\left\{ \frac{(1-\lambda\gamma\beta)[H-N_1k]-\lambda N_1 k \alpha \gamma \beta}{N_1 k \alpha \gamma \beta} \right\}}{\left\{ \frac{N_1 k \alpha \gamma \beta - (1-\gamma\beta)[H-N_1k]}{N_1 k \alpha \gamma \beta} \right\}} m_{11}^{\lambda\gamma} [y_3 + y_1]^\lambda \quad (\text{A.47})$$

Rearranging it, we have:

$$m_{21} = \left[ \frac{H(1-\beta\lambda\gamma) - [1 - (1-\alpha)\beta\lambda\gamma]kN_1}{[1 - (1-\alpha)\beta\gamma]kN_1 - (1-\beta\gamma)H} \right]^{\frac{1}{\gamma}} m_{11}^\lambda \left( \frac{[y_3 + y_1]^\lambda}{y_2} \right)^{\frac{1}{\gamma}} \quad (\text{A.48})$$

Substituting  $m_{11}$ , we have:

$$m_{21} = \left\{ \begin{array}{l} \left[ \frac{H(1-\beta\lambda\gamma) - [1 - (1-\alpha)\beta\lambda\gamma]kN_1}{[1 - (1-\alpha)\beta\gamma]kN_1 - (1-\beta\gamma)H} \right]^{\frac{1}{\gamma}} \left( \frac{[y_3 + y_1]^\lambda}{y_2} \right)^{\frac{1}{\gamma}} \times \\ \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ \frac{[1 - (1-\alpha)\beta\gamma]kN_1 - (1-\beta\gamma)H}{[1 - (1-\alpha)\beta\gamma]kN_2 - (1-\beta\gamma)H} \right]^{\beta-1} \right\}^{\frac{\lambda}{\lambda\gamma\beta-1}} \\ \times \frac{M_1}{\left[ N_2 + N_1 \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ \frac{[1 - (1-\alpha)\beta\gamma]kN_1 - (1-\beta\gamma)H}{[1 - (1-\alpha)\beta\gamma]kN_2 - (1-\beta\gamma)H} \right]^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}} \right]^\lambda (M_1)^\lambda} \end{array} \right\} \quad (\text{A.49})$$

Then, also notice that:

$$m_{12} = \frac{M_1}{N_2 + N_1 \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ \frac{[1 - (1-\alpha)\beta\gamma]kN_1 - (1-\beta\gamma)H}{[1 - (1-\alpha)\beta\gamma]kN_2 - (1-\beta\gamma)H} \right]^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}}} \quad (\text{A.50})$$

and:

$$m_{32} = \frac{M_3}{\left[ N_2 + N_1 \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ \frac{[1 - (1-\alpha)\beta\gamma]kN_1 - (1-\beta\gamma)H}{[1 - (1-\alpha)\beta\gamma]kN_2 - (1-\beta\gamma)H} \right]^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}} \right]} \quad (\text{A.51})$$

and

$$m_{22} = \left\{ \begin{array}{l} \left[ \frac{H(1-\beta\lambda\gamma) - [1 - (1-\alpha)\beta\lambda\gamma]kN_2}{[1 - (1-\alpha)\beta\gamma]kN_2 - (1-\beta\gamma)H} \right]^{\frac{1}{\gamma}} \left( \frac{[y_3 + y_1]^\lambda}{y_2} \right)^{\frac{1}{\gamma}} \times \\ \left( \frac{M_1}{N_2 + N_1 \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ \frac{[1 - (1-\alpha)\beta\gamma]kN_1 - (1-\beta\gamma)H}{[1 - (1-\alpha)\beta\gamma]kN_2 - (1-\beta\gamma)H} \right]^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}}} \right)^\lambda \end{array} \right\} \quad (\text{A.52})$$

**Proposition A. 1** *If  $\lambda > 1$ ,  $\lambda\gamma < 1$ , and  $\lambda\gamma\beta < 1$ , there is no equilibrium in which  $A_2 > A_1$ , and  $m_{i1} \geq m_{i2}$ .*

Assume  $A_2 > A_1$ . Before we continue, we prove the following Lemma:

**Lemma A. 1** *If  $m_{11} > m_{12}$ , then  $p_1 > p_2$ ,  $N_2 > N_1$ , and  $m_{21} > m_{22}$*

**Proof.** Going back to the system, we have:

$$\left\{ \begin{array}{l}
A_1 \left[ \frac{\left[ m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda \right]^{\beta-1} \times [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^{\lambda-1}}{\left[ m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right]^{\beta-1} \times [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^{\lambda-1}} \right] m_{11}^{\gamma-1} = \left( \frac{p_1}{p_2} \right)^\alpha A_2 m_{12}^{\gamma-1} \quad (1') \\
A_1 \left[ \frac{\left[ m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda \right]^{\beta-1}}{\left[ m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right]^{\beta-1}} \right] m_{21}^{\gamma-1} = \left( \frac{p_1}{p_2} \right)^\alpha A_2 \left[ m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right]^{\beta-1} m_{22}^{\gamma-1} \quad (2') \\
A_1 \left[ \frac{\left[ m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda \right]^{\beta-1} \times [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^{\lambda-1}}{\left[ m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right]^{\beta-1} \times [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^{\lambda-1}} \right] m_{31}^{\gamma-1} = \left( \frac{p_1}{p_2} \right)^\alpha A_2 m_{32}^{\gamma-1} \quad (3') \\
N_1 m_{i1} + N_2 m_{i2} = M_i, \forall i \in \{1, 2, 3\} \quad (4', 5', 6') \\
\left[ m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda \right]^{\beta-1} \left\{ \lambda [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda + m_{21}^\gamma y_2 \right\} = \left[ \frac{H}{N_1} - k \right] \frac{p_1}{\alpha \gamma \beta A_1} \quad (7') \\
\left[ m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right]^{\beta-1} \left\{ \lambda [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda + m_{22}^\gamma y_2 \right\} = \left[ \frac{H}{N_2} - k \right] \frac{p_2}{\alpha \gamma \beta A_2} \quad (8') \\
\left[ m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda \right]^{\beta-1} \left\{ (1 - \lambda \gamma \beta) [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda + (1 - \gamma \beta) m_{21}^\gamma y_2 \right\} = \frac{k}{A_1} p_1 \quad (9') \\
\left[ m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right]^{\beta-1} \left\{ (1 - \lambda \gamma \beta) [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda + (1 - \gamma \beta) m_{22}^\gamma y_2 \right\} = \frac{k}{A_2} p_2 \quad (10')
\end{array} \right. \quad (A.53)$$

From the last two equations, we have:

$$\left[ \begin{array}{l}
\left[ m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda \right]^{\beta-1} \times \\
\times \left\{ (1 - \lambda \gamma \beta) [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda + (1 - \gamma \beta) m_{21}^\gamma y_2 \right\} \\
\left[ m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right]^{\beta-1} \times \\
\times \left\{ (1 - \lambda \gamma \beta) [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda + (1 - \gamma \beta) m_{22}^\gamma y_2 \right\}
\end{array} \right] \frac{p_2}{p_1} \frac{A_1}{A_2} = \frac{k}{A_2} p_2 \quad (A.54)$$

Equating this two expressions, we have:

$$\left[ \begin{array}{l}
\left[ \left[ m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda \right]^{\beta-1} \times \right. \\
\left. \times \left\{ (1 - \lambda \gamma \beta) [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda + (1 - \gamma \beta) m_{21}^\gamma y_2 \right\} \right] \frac{p_2}{p_1} \frac{A_1}{A_2} \\
- \left[ \left[ m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right]^{\beta-1} \times \right. \\
\left. \times \left\{ (1 - \lambda \gamma \beta) [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda + (1 - \gamma \beta) m_{22}^\gamma y_2 \right\} \right]
\end{array} \right] = 0 \quad (A.55)$$

since  $M_1 = M_3$ , we have that  $m_{31} = m_{11}$  and  $m_{32} = m_{12}$ . Based on these results, we have:

$$\left[ \begin{array}{l}
\left[ m_{21}^\gamma y_2 + m_{11}^{\lambda \gamma} [y_3 + y_1]^\lambda \right]^{\beta-1} \left\{ (1 - \lambda \gamma \beta) m_{11}^{\lambda \gamma} [y_3 + y_1]^\lambda + (1 - \gamma \beta) m_{21}^\gamma y_2 \right\} \frac{p_2}{p_1} \frac{A_1}{A_2} \\
- \left[ m_{22}^\gamma y_2 + m_{12}^{\lambda \gamma} [y_3 + y_1]^\lambda \right]^{\beta-1} \left\{ (1 - \lambda \gamma \beta) m_{12}^{\lambda \gamma} [y_3 + y_1]^\lambda + (1 - \gamma \beta) m_{22}^\gamma y_2 \right\}
\end{array} \right] = 0 \quad (A.56)$$

Then, from equation (1), we have - again using symmetry:

$$\left[ m_{21}^\gamma y_2 + m_{11}^{\lambda\gamma} [y_3 + y_1]^\lambda \right]^{\beta-1} = \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} \left[ m_{22}^\gamma y_2 + m_{12}^{\lambda\gamma} [y_3 + y_1]^\lambda \right]^{\beta-1} \frac{m_{12}^{\lambda\gamma-1}}{m_{11}^{\lambda\gamma-1}} \quad (\text{A.57})$$

Substituting it back, we have:

$$\left[ m_{22}^\gamma y_2 + m_{12}^{\lambda\gamma} [y_3 + y_1]^\lambda \right]^{\beta-1} \left[ \begin{aligned} & (1 - \lambda\gamma\beta) [y_3 + y_1]^\lambda \left[ \left( \frac{p_2}{p_1} \right)^{1-\alpha} m_{11} - m_{12} \right] m_{12}^{\lambda\gamma-1} \\ & + (1 - \gamma\beta) y_2 \left[ \left( \frac{p_2}{p_1} \right)^{1-\alpha} \left( \frac{m_{12}}{m_{11}} \right)^{\lambda\gamma-1} m_{21}^\gamma - m_{22}^\gamma \right] \end{aligned} \right] = 0 \quad (\star) \quad (\text{A.58})$$

Since:

$$\begin{aligned} \left( \frac{m_{11}}{m_{12}} \right)^{\lambda\gamma-1} &= \left( \frac{m_{21}}{m_{22}} \right)^{\gamma-1} \\ &\Downarrow \\ \left( \frac{m_{12}}{m_{11}} \right)^{\lambda\gamma-1} &= \left( \frac{m_{22}}{m_{21}} \right)^{\gamma-1} \end{aligned}$$

we have:

$$\left[ m_{22}^\gamma y_2 + m_{12}^{\lambda\gamma} [y_3 + y_1]^\lambda \right]^{\beta-1} \left\{ \begin{aligned} & (1 - \lambda\gamma\beta) [y_3 + y_1]^\lambda \left[ \left( \frac{p_2}{p_1} \right)^{1-\alpha} \frac{m_{11}}{m_{12}} - 1 \right] m_{12}^{\lambda\gamma} \\ & + (1 - \gamma\beta) y_2 \left[ \left( \frac{p_2}{p_1} \right)^{1-\alpha} \frac{m_{21}}{m_{22}} - 1 \right] m_{22}^\gamma \end{aligned} \right\} = 0 \quad (\text{A.59})$$

Since:

$$\frac{m_{21}}{m_{22}} = \left( \frac{m_{11}}{m_{12}} \right)^{\frac{\lambda\gamma-1}{\gamma-1}} \quad (\text{A.60})$$

we have:

$$\left[ m_{22}^\gamma y_2 + m_{12}^{\lambda\gamma} [y_3 + y_1]^\lambda \right]^{\beta-1} \left\{ \begin{aligned} & (1 - \lambda\gamma\beta) [y_3 + y_1]^\lambda \left[ \left( \frac{p_2}{p_1} \right)^{1-\alpha} \frac{m_{11}}{m_{12}} - 1 \right] m_{12}^{\lambda\gamma} \\ & (1 - \gamma\beta) y_2 \left[ \left( \frac{p_2}{p_1} \right)^{1-\alpha} \left( \frac{m_{11}}{m_{12}} \right)^{\frac{\lambda\gamma-1}{\gamma-1}} - 1 \right] m_{22}^\gamma \end{aligned} \right\} = 0 \quad (\text{A.61})$$

Assuming that  $\lambda\gamma\beta < 1$ , we have that the only terms that can be negative are the ones inside the squared brackets inside the curly brackets.

Since  $\frac{\lambda\gamma-1}{\gamma-1} \in (0, 1)$  :

$$\begin{aligned} & \left[ \left( \frac{p_2}{p_1} \right)^{1-\alpha} \frac{m_{11}}{m_{12}} - 1 \right] - \left[ \left( \frac{p_2}{p_1} \right)^{1-\alpha} \left( \frac{m_{11}}{m_{12}} \right)^{\frac{\lambda\gamma-1}{\gamma-1}} - 1 \right] = \\ & = \left( \frac{p_2}{p_1} \right)^{1-\alpha} \frac{m_{11}}{m_{12}} \left[ 1 - \left( \frac{m_{12}}{m_{11}} \right)^{\frac{\gamma(\lambda-1)}{1-\gamma}} \right] \end{aligned}$$

Since  $\frac{\gamma(\lambda-1)}{1-\gamma} \in (0, 1)$ , the sign will depend on  $\frac{m_{12}}{m_{11}}$ . Since

$$\frac{m_{12}}{m_{11}} < 1 \Rightarrow \left[ \left( \frac{p_2}{p_1} \right)^{1-\alpha} \frac{m_{11}}{m_{12}} - 1 \right] > \left[ \left( \frac{p_2}{p_1} \right)^{1-\alpha} \left( \frac{m_{11}}{m_{12}} \right)^{\frac{\lambda\gamma-1}{\gamma-1}} - 1 \right].$$

In order to keep the equality, we must have:

$$\begin{aligned} \left( \frac{p_2}{p_1} \right)^{1-\alpha} \frac{m_{11}}{m_{12}} - 1 &> 0 \\ \frac{m_{11}}{m_{12}} &> \left( \frac{p_1}{p_2} \right)^{1-\alpha} \end{aligned}$$

and

$$\begin{aligned} \left( \frac{p_2}{p_1} \right)^{1-\alpha} \left( \frac{m_{11}}{m_{12}} \right)^{\frac{\lambda\gamma-1}{\gamma-1}} - 1 &< 0 \\ \left( \frac{m_{11}}{m_{12}} \right)^{\frac{1-\lambda\gamma}{1-\gamma}} &< \left( \frac{p_1}{p_2} \right)^{1-\alpha} \end{aligned}$$

since  $\frac{1-\lambda\gamma}{1-\gamma} \in (0, 1)$  and  $\alpha \in (0, 1)$ , we have that:

$$\left( \frac{p_1}{p_2} \right)^{1-\alpha} \in \left( \left( \frac{m_{11}}{m_{12}} \right)^{\frac{1-\lambda\gamma}{1-\gamma}}, \frac{m_{11}}{m_{12}} \right) \quad (\text{A.62})$$

since  $\frac{m_{11}}{m_{12}} > 1$ , we have that  $p_1 > p_2$ .

We also showed earlier that:

$$\frac{m_{21}}{m_{22}} = \left( \frac{m_{11}}{m_{12}} \right)^{\frac{1-\lambda\gamma}{1-\gamma}} \quad (\text{A.63})$$

since  $\frac{m_{11}}{m_{12}} > 1$ , we have that  $m_{21} > m_{22}$ .

Finally, from equations (7') and (8'), we have:

$$\begin{aligned} N_1 &= \frac{H}{\left\{ \frac{\alpha\gamma\beta A_1}{p_1} [m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda]^{\beta-1} \{ \lambda [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda + m_{21}^\gamma y_2 \} + k \right\}} \\ N_2 &= \frac{H}{\left\{ \frac{\alpha\gamma\beta A_2}{p_2} [m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda]^{\beta-1} \{ \lambda [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda + m_{22}^\gamma y_2 \} + k \right\}} \end{aligned} \quad (\text{A.64})$$

Since, from (9') and (10'):

$$\begin{aligned} \frac{A_1}{p_1} &= \frac{k}{\left[ \begin{aligned} &[m_{21}^\gamma y_2 + [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda]^{\beta-1} \times \\ &\times \left\{ (1-\beta\gamma) m_{21}^\gamma y_2 + (1-\lambda\gamma\beta) [m_{31}^\gamma y_3 + m_{11}^\gamma y_1]^\lambda \right\} \end{aligned} \right]} \\ \frac{A_2}{p_2} &= \frac{k}{\left[ \begin{aligned} &[m_{22}^\gamma y_2 + [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda]^{\beta-1} \times \\ &\times \left\{ (1-\beta\gamma) m_{22}^\gamma y_2 + (1-\lambda\gamma\beta) [m_{32}^\gamma y_3 + m_{12}^\gamma y_1]^\lambda \right\} \end{aligned} \right]} \end{aligned}$$

Substituting it back, we have:

$$N_1 = \frac{H}{k} \frac{(1-\beta\gamma) m_{21}^\gamma y_2 + (1-\lambda\gamma\beta) m_{11}^{\lambda\gamma} [y_3 + y_1]^\lambda}{[1 - (1-\alpha)\beta\gamma] m_{21}^\gamma y_2 + [1 - (1-\alpha)\lambda\gamma\beta] m_{11}^{\lambda\gamma} [y_3 + y_1]^\lambda} \quad (\text{A.65})$$

and

$$N_2 = \frac{H}{k} \frac{(1-\beta\gamma) m_{22}^\gamma y_2 + (1-\lambda\gamma\beta) m_{12}^{\lambda\gamma} [y_3 + y_1]^\lambda}{[1 - (1-\alpha)\beta\gamma] m_{22}^\gamma y_2 + [1 - (1-\alpha)\lambda\gamma\beta] m_{12}^{\lambda\gamma} [y_3 + y_1]^\lambda} \quad (\text{A.66})$$

Then:

$$\frac{N_1}{N_2} = 1 + \frac{\alpha\beta\gamma y_2 [y_3 + y_1]^\lambda (\lambda - 1) \left[ \left( \frac{m_{12}}{m_{11}} \right)^{\frac{(\lambda-1)\gamma}{1-\gamma}} - 1 \right] m_{22}^\gamma m_{11}^{\lambda\gamma}}{\left[ \begin{array}{l} [1 - (1-\alpha)\beta\gamma] (1-\beta\gamma) (y_2)^2 m_{22}^\gamma m_{21}^\gamma \\ + [1 - (1-\alpha)\beta\gamma] (1-\lambda\gamma\beta) y_2 [y_3 + y_1]^\lambda m_{21}^\gamma m_{12}^{\lambda\gamma} \\ + [1 - (1-\alpha)\lambda\gamma\beta] (1-\beta\gamma) [y_3 + y_1]^\lambda y_2 m_{22}^\gamma m_{11}^{\lambda\gamma} \\ + [1 - (1-\alpha)\lambda\gamma\beta] (1-\lambda\gamma\beta) [y_3 + y_1]^{2\lambda} m_{11}^{\lambda\gamma} m_{12}^{\lambda\gamma} \end{array} \right]} \quad (\star\star) \quad (\text{A.67})$$

Since  $\frac{m_{12}}{m_{11}} < 1$ , we have that  $\frac{N_1}{N_2} < 1 \Rightarrow N_1 < N_2$ .

Then, back in the system, rearranging it, we have:

$$\left\{ \left[ \left( \frac{M_1}{N_2 + N_1 \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} Z^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}}} \right)^\lambda \left( \frac{[y_3 + y_1]^\lambda}{y_2} \right)^{\frac{1}{\gamma}} \times \right. \right. \right. \quad (1'') \\ \left. \left. \times \left[ N_1 \left\{ \left[ \frac{H(1-\beta\lambda\gamma) - [1 - (1-\alpha)\beta\lambda\gamma] k N_1}{[1 - (1-\alpha)\beta\gamma] k N_1 - (1-\beta\gamma) H} \right]^{\frac{1}{\gamma}} \times \right. \right. \right. \right. \quad (1'') \\ \left. \left. \left. \times \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} Z^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}} \right\} \right] \right. \right. \quad (1'') \\ \left. \left. \left. + N_2 \left[ \frac{H(1-\beta\lambda\gamma) - [1 - (1-\alpha)\beta\lambda\gamma] k N_2}{[1 - (1-\alpha)\beta\gamma] k N_2 - (1-\beta\gamma) H} \right]^{\frac{1}{\gamma}} \right] \right] \right\} = M_2 \quad (1'') \\ \left. \left. \left. \left[ \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} Z^{\beta-1} \right\}^{\frac{\lambda-1}{1-\lambda\gamma\beta}} \times \right. \right. \right. \quad (2'') \\ \left. \left. \left. \times Z^{\frac{\gamma-1}{\gamma}} \right] \right] \right\} = 1 \quad (2'') \quad (\text{A.68}) \\ \left. \left. \left. \left[ \left[ \frac{[H - (1-\alpha)k N_1] \beta\gamma (\lambda-1)}{H(1-\beta\gamma) - [1 - (1-\alpha)\beta\gamma] k N_1} \right]^{\beta-1} \left[ \frac{(\lambda-1)}{H(1-\beta\gamma) - [1 - (1-\alpha)\beta\gamma] k N_1} \right] \times \right. \right. \right. \quad (3'') \\ \left. \left. \left. \times \left( \frac{\left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} Z^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}} M_1}{\left[ N_2 + N_1 \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} Z^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}} \right]} \right) \right] \right. \right. \quad (3'') \\ \left. \left. \left. \left. \right] [y_3 + y_1]^{\lambda\beta} \right] \right\} = \frac{1}{N_1} \frac{p_1}{\alpha\gamma\beta A_1} \quad (3'') \\ \left. \left. \left. \left( \left[ \frac{(\lambda-1)}{H(1-\beta\gamma) - [1 - (1-\alpha)\beta\gamma] k N_2} \right] \left[ \frac{[H - (1-\alpha)k N_2] \beta\gamma (\lambda-1)}{H(1-\beta\gamma) - [1 - (1-\alpha)\beta\gamma] k N_2} \right]^{\beta-1} \times \right. \right. \right. \quad (4'') \\ \left. \left. \left. \times \left( \frac{M_1}{N_2 + N_1 \left\{ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A_2}{A_1} Z^{\beta-1} \right\}^{\frac{1}{\lambda\gamma\beta-1}}} \right) \right] \right. \right. \quad (4'') \\ \left. \left. \left. \left. \right] [y_3 + y_1]^{\lambda\beta} \right] \right\} = \frac{1}{N_2} \frac{p_2}{\alpha\gamma\beta A_2} \quad (4'')$$

where  $Z = \frac{[1 - (1-\alpha)\beta\gamma] k N_1 - (1-\beta\gamma) H}{[1 - (1-\alpha)\beta\gamma] k N_2 - (1-\beta\gamma) H} \times \frac{(1-\alpha)k N_2 - H}{(1-\alpha)k N_1 - H}$

Then, from (3'') and (4''), we have:

$$\left\{ \left( \frac{m_{11}}{m_{12}} \right)^{\lambda\gamma\beta} \left[ \frac{H - (1-\alpha)k N_1}{H - (1-\alpha)k N_2} \right]^{\beta-1} \right\} \frac{N_1}{N_2} = \frac{A_2 p_1}{A_1 p_2} \quad (\text{A.69})$$

once

$$\frac{m_{11}}{m_{12}} = \left\{ \left( \frac{p_2}{p_1} \right)^\alpha \frac{A_1}{A_2} \left[ \frac{(1-\beta\gamma)H - [1-(1-\alpha)\beta\gamma]kN_2}{(1-\beta\gamma)H - [1-(1-\alpha)\beta\gamma]kN_1} \right]^{\beta-1} \times \left[ \frac{H-(1-\alpha)kN_1}{H-(1-\alpha)kN_2} \right]^{\beta-1} \right\}^{\frac{1}{1-\lambda\gamma\beta}} \quad (\text{A.70})$$

Substituting it back, we have:

$$\left[ \left[ \frac{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_2}{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_1} \right]^{\frac{\beta(1-\lambda\gamma)}{1-\lambda\gamma\beta}} \times \frac{H-(1-\alpha)kN_1}{H-(1-\alpha)kN_2} \times \left[ \frac{H-(1-\alpha)kN_2}{H-(1-\alpha)kN_1} \right] \right]^{\frac{1-\lambda\gamma\beta}{1-\lambda\gamma\beta}} \frac{N_1}{N_2} = \left( \frac{A_2}{A_1} \right)^{\frac{1}{1-\lambda\gamma\beta}} \left( \frac{p_1}{p_2} \right)^{\frac{1-(1-\alpha)\lambda\gamma\beta}{1-\lambda\gamma\beta}} \quad (\star\star\star) \quad (\text{A.71})$$

Notice that:

$$\left[ \frac{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_2}{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_1} \times \frac{H-(1-\alpha)kN_1}{H-(1-\alpha)kN_2} \right] - 1 = \frac{Hk\alpha(N_1 - N_2)}{\left\{ [H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_1] \times [H-(1-\alpha)kN_2] \right\}} \quad (\text{A.72})$$

Since:

$$m_{21} = \left[ -\frac{H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_1}{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_1} \right]^{\frac{1}{\gamma}} m_{11}^\lambda \left( \frac{[y_3 + y_1]^\lambda}{y_2} \right)^{\frac{1}{\gamma}} \quad (\text{A.73})$$

and  $m_{21} > 0$ , we must have that:

$$-\frac{H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_1}{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_1} > 0 \quad (\text{A.74})$$

Since  $H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_1$  is decreasing in  $\lambda$ , we must have:

$$H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_1 > 0 \quad (\text{A.75})$$

and

$$H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_1 < 0 \quad (\text{A.76})$$

but them, we have that:

$$\left\{ \begin{array}{l} [H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_1] \times \\ \times [H-(1-\alpha)kN_2] \end{array} \right\} > 0 \quad (\text{A.77})$$

Since  $N_1 < N_2$ , this implies that:

$$\left[ \frac{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_2}{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_1} \times \frac{H-(1-\alpha)kN_1}{H-(1-\alpha)kN_2} \right] < 1 \quad (\text{A.78})$$

Then, from (★★★), we have:

$$\begin{aligned} \left[ \begin{array}{l} \frac{H(1-\beta\gamma)-[1-(1-\alpha)\beta\gamma]kN_2}{H(1-\beta\gamma)-[1-(1-\alpha)\beta\gamma]kN_1} \\ \times \frac{H-(1-\alpha)kN_1}{H-(1-\alpha)kN_2} \end{array} \right]^{\frac{\beta(1-\lambda\gamma)}{1-\lambda\gamma\beta}} &< 1 \\ \left[ \frac{H-(1-\alpha)kN_2}{H-(1-\alpha)kN_1} \right] &< 1 \\ \frac{N_1}{N_2} &< 1 \end{aligned}$$

Therefore  $LHS < 1$ . We also know that  $\left(\frac{p_1}{p_2}\right)^{\frac{1-(1-\alpha)\lambda\gamma\beta}{1-\lambda\gamma\beta}} > 1$ . Given  $A_2 > A_1$ ,  $RHS > 1$  and we have a contradiction

In order to complete our proof, assume that  $\frac{m_{11}}{m_{12}} = 1 \Rightarrow m_{11} = m_{12}$ . Given that  $\frac{m_{21}}{m_{22}} = \left(\frac{m_{11}}{m_{12}}\right)^{\frac{1-\lambda\gamma}{1-\gamma}}$ , we have that  $m_{21} = m_{22}$ . Then, from (★★) we have  $N_1 = N_2$  and from (★) we have  $p_1 = p_2$ . But, combining these results and (★★★), we again have a contradiction, once  $LHS = 1$  while  $RHS > 1$  once  $A_2 > A_1$ . ■

**Corollary A. 1** *There is no equilibrium in which  $A_1 > A_2$  and  $m_{i2} > m_{i1}$ ,  $\forall i \in \{1, 2, 3\}$ .*

**Theorem A. 1** *City Size and TFP. Let  $A_1 > A_2$ ,  $\beta > 1$ ,  $\lambda\gamma\beta < 1$ , and  $\gamma < 1$ . Then the more productive city is larger,  $S_1 > S_2$ .*

**Proof.** Before we start, define:

$$Z = \left(\frac{p_1}{p_2}\right)^\alpha \frac{A_2}{A_1} \left[ \begin{array}{l} \frac{[1-(1-\alpha)\beta\gamma]kN_1-(1-\beta\gamma)H}{[1-(1-\alpha)\beta\gamma]kN_2-(1-\beta\gamma)H} \\ \times \frac{(1-\alpha)kN_2-H}{(1-\alpha)kN_1-H} \end{array} \right]^{\beta-1} \quad (\text{A.79})$$

Notice that:

$$\frac{m_{11}}{m_{12}} = Z^{\frac{1}{\lambda\gamma\beta-1}} \quad (\text{A.80})$$

Since  $A_1 > A_2$ , from Corollary 1 we have  $m_{11} > m_{12}$ . From Lemma 1 and  $\lambda\gamma\beta < 1$ , we have that  $Z < 1$ .

Notice that:

$$\begin{aligned} S_1 &= N_1 (2 * m_{11} + m_{21}) \\ S_1 &= N_2 \left( \begin{array}{l} 2 * \frac{N_1}{N_2} \frac{Z^{\frac{1}{\lambda\gamma\beta-1}}}{\left[N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}\right]} M_1 \\ + \frac{N_1}{N_2} \left\{ \begin{array}{l} \left[ \frac{H(1-\beta\lambda\gamma)-[1-(1-\alpha)\beta\lambda\gamma]kN_1}{[1-(1-\alpha)\beta\gamma]kN_1-(1-\beta\gamma)H} \right]^{\frac{1}{\gamma}} \\ \times \left( \frac{[y_3+y_1]^\lambda}{y_2} \right)^{\frac{1}{\gamma}} \left( \frac{Z^{\frac{1}{\lambda\gamma\beta-1}}}{\left[N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}\right]} M_1 \right)^\lambda \end{array} \right\} \end{array} \right) \end{aligned}$$

while:

$$\begin{aligned}
S_2 &= N_2 (2 * m_{12} + m_{22}) \\
&= N_2 \left( 2 * \frac{M_1}{N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}} + \left\{ \left[ \frac{H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_2}{[1-(1-\alpha)\beta\gamma]kN_2 - (1-\beta\gamma)H} \right]^{\frac{1}{\gamma}} \left( \frac{[y_3+y_1]^\lambda}{y_2} \right)^{\frac{1}{\gamma}} \left( \frac{M_1}{N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}} \right)^\lambda \right\} \right)
\end{aligned}$$

then:

$$S_2 - S_1 = N_2 \left\{ \begin{aligned} & 2 \frac{\left[ 1 - \frac{N_1}{N_2} Z^{\frac{1}{\lambda\gamma\beta-1}} \right] M_1}{N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}} + \\ & + \left[ \left[ 1 - \frac{N_1}{N_2} Z^{\frac{1}{\lambda\gamma\beta-1}} \right] \left[ \frac{H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_2}{[1-(1-\alpha)\beta\gamma]kN_2 - (1-\beta\gamma)H} \right]^{\frac{1}{\gamma}} \times \right. \\ & \left. \times \left( \frac{[y_3+y_1]^\lambda}{y_2} \right)^{\frac{1}{\gamma}} \left( \frac{M_1}{N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}} \right)^\lambda \right] \end{aligned} \right\} \quad (\text{A.81})$$

Since:

$$\left\{ \begin{aligned} & \left[ \frac{H - (1-\alpha)kN_1}{H - (1-\alpha)kN_2} \right]^{\beta-1} \\ & \times \left[ \frac{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_2}{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_1} \right]^\beta \end{aligned} \right\} Z^{\frac{\lambda\gamma\beta}{\lambda\gamma\beta-1}} = \frac{N_2 p_1 A_2}{N_1 p_2 A_1} \quad (\text{A.82})$$

Then, from  $\frac{(3'')}{(4'')}$ , we have:

$$\frac{N_1}{N_2} = \frac{\frac{p_1 A_2}{p_2 A_1}}{\left[ \frac{H - (1-\alpha)kN_1}{H - (1-\alpha)kN_2} \right]^{\beta-1} \left[ \frac{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_2}{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_1} \right]^\beta Z^{\frac{\lambda\gamma\beta}{\lambda\gamma\beta-1}}} \quad (\text{A.83})$$

so:

$$\begin{aligned}
\frac{N_1}{N_2} Z^{\frac{1}{\lambda\gamma\beta-1}} &= \frac{\frac{p_1 A_2}{p_2 A_1}}{\left[ \frac{H - (1-\alpha)kN_1}{H - (1-\alpha)kN_2} \right]^{\beta-1} \left[ \frac{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_2}{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_1} \right]^\beta} \left( \frac{1}{Z} \right) \\
&= \left( \frac{p_1}{p_2} \right)^{1-\alpha} \left[ \frac{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_1}{H(1-\beta\gamma) - [1-(1-\alpha)\beta\gamma]kN_2} \right] > 1
\end{aligned}$$

Therefore, since  $\frac{N_1}{N_2} Z^{\frac{1}{\lambda\gamma\beta-1}} > 1$ , we have that  $\left[ 1 - \frac{N_1}{N_2} Z^{\frac{1}{\lambda\gamma\beta-1}} \right] < 0$ . Since  $\lambda > 1$  and  $\frac{N_1}{N_2} Z^{\frac{\lambda}{\lambda\gamma\beta-1}} \cdot \frac{N_1}{N_2} Z^{\frac{1}{\lambda\gamma\beta-1}}$ , we also have that  $\left[ 1 - \frac{N_1}{N_2} Z^{\frac{\lambda}{\lambda\gamma\beta-1}} \right] < 0$ . Therefore:

$$S_2 - S_1 < 0 \quad (\text{A.84})$$

and we have that the city with the highest TFP is also the largest city. ■

**Theorem A. 2** *Thick tails.* Let  $A_1 > A_2, \beta > 1, \lambda > 1$ , and  $\lambda\gamma\beta < 1$ , the skill distribution in the larger city has thicker tails.

**Proof.** Consider the distributions, denoted by  $pdf_{ij}$  :



$$pdf_{11} = \frac{N_1 m_{11}}{S_1} = \frac{\frac{Z^{\frac{1}{\lambda\gamma\beta-1}}}{[N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}]} M_1}{\left( 2 * \frac{Z^{\frac{1}{\lambda\gamma\beta-1}}}{[N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}]} M_1 + \left\{ \left[ \frac{H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_1}{[1-(1-\alpha)\beta\gamma]kN_1 - (1-\beta\gamma)H} \right]^{\frac{1}{\gamma}} \times \left( \frac{[y_3 + y_1]^\lambda}{y_2} \right)^{\frac{1}{\gamma}} \left( \frac{Z^{\frac{1}{\lambda\gamma\beta-1}}}{[N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}]} M_1 \right)^\lambda \right\} \right)^\lambda} \quad (A.85)$$

$$pdf_{12} = \frac{\frac{M_1}{N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}}}{\left( 2 * \frac{M_1}{N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}} + \left\{ \left[ \frac{H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_2}{[1-(1-\alpha)\beta\gamma]kN_2 - (1-\beta\gamma)H} \right]^{\frac{1}{\gamma}} \times \left( \frac{[y_3 + y_1]^\lambda}{y_2} \right)^{\frac{1}{\gamma}} \left( \frac{M_1}{N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}} \right)^\lambda \right\} \right)^\lambda} \quad (A.86)$$

Since:

$$\begin{aligned} & \left( \frac{M_1}{N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}} \right)^\lambda \left( \frac{[y_3 + y_1]^\lambda}{y_2} \right)^{\frac{1}{\gamma}} \\ &= \frac{M_2}{\left[ N_1 \left[ \frac{H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_1}{[1-(1-\alpha)\beta\gamma]kN_1 - (1-\beta\gamma)H} \right]^{\frac{1}{\gamma}} Z^{\frac{\lambda}{\lambda\gamma\beta-1}} + N_2 \left[ \frac{H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_2}{[1-(1-\alpha)\beta\gamma]kN_2 - (1-\beta\gamma)H} \right]^{\frac{1}{\gamma}} \right]} \end{aligned}$$

we have:

$$pdf_{11} = \frac{Z^{\frac{1}{\lambda\gamma\beta-1}} M_1}{\left( \frac{2Z^{\frac{1}{\lambda\gamma\beta-1}} M_1}{M_2 [N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}]} + \left[ N_1 Z^{\frac{\lambda}{\lambda\gamma\beta-1}} + N_2 \left[ \frac{H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_2}{[1-(1-\alpha)\beta\gamma]kN_2 - (1-\beta\gamma)H} \right]^{\frac{1}{\gamma}} \right] \times \frac{[1-(1-\alpha)\beta\gamma]kN_1 - (1-\beta\gamma)H}{H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_1} \right)^{\frac{1}{\gamma}} \right)^\lambda} \quad (A.87)$$

and

$$pdf_{12} = \frac{M_1}{\left( \frac{2M_1}{M_2 [N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}}]} + \left[ N_1 Z^{\frac{\lambda}{\lambda\gamma\beta-1}} \left[ \frac{H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_1}{[1-(1-\alpha)\beta\gamma]kN_1 - (1-\beta\gamma)H} \right]^{\frac{1}{\gamma}} \right] \times \frac{[1-(1-\alpha)\beta\gamma]kN_2 - (1-\beta\gamma)H}{H(1-\beta\lambda\gamma) - [1-(1-\alpha)\beta\lambda\gamma]kN_2} + N_2 \right)^{\frac{1}{\gamma}} \right)^\lambda} \quad (A.88)$$

$$\frac{pdf_{11}}{pdf_{12}} = \frac{\left( \left[ \begin{array}{l} 2Z^{\frac{1}{\lambda\gamma\beta-1}} M_1 \times \left[ \begin{array}{l} \frac{H(1-\beta\lambda\gamma)-[1-(1-\alpha)\beta\lambda\gamma]kN_2}{H(1-\beta\lambda\gamma)-[1-(1-\alpha)\beta\lambda\gamma]kN_1} \\ \times \frac{H(1-\beta\gamma)-[1-(1-\alpha)\beta\gamma]kN_1}{H(1-\beta\gamma)-[1-(1-\alpha)\beta\gamma]kN_2} \end{array} \right]^{\frac{1}{\gamma}} \times \\ \times \left[ \begin{array}{l} N_1 Z^{\frac{\lambda}{\lambda\gamma\beta-1}} \left[ \begin{array}{l} \frac{H(1-\beta\lambda\gamma)-[1-(1-\alpha)\beta\lambda\gamma]kN_1}{H(1-\beta\lambda\gamma)-[1-(1-\alpha)\beta\lambda\gamma]kN_2} \\ \times \frac{H(1-\beta\gamma)-[1-(1-\alpha)\beta\gamma]kN_2}{H(1-\beta\gamma)-[1-(1-\alpha)\beta\gamma]kN_1} \end{array} \right]^{\frac{1}{\gamma}} \\ + N_2 \end{array} \right] \\ + M_2 \left[ N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}} \right] Z^{\frac{1}{\lambda\gamma\beta-1}} \end{array} \right]}{\left( \left[ \begin{array}{l} 2Z^{\frac{1}{\lambda\gamma\beta-1}} M_1 \times \left[ \begin{array}{l} \frac{H(1-\beta\lambda\gamma)-[1-(1-\alpha)\beta\lambda\gamma]kN_2}{H(1-\beta\lambda\gamma)-[1-(1-\alpha)\beta\lambda\gamma]kN_1} \\ \times \frac{H(1-\beta\gamma)-[1-(1-\alpha)\beta\gamma]kN_1}{H(1-\beta\gamma)-[1-(1-\alpha)\beta\gamma]kN_2} \end{array} \right]^{\frac{1}{\gamma}} \times \\ \times \left[ \begin{array}{l} N_1 Z^{\frac{\lambda}{\lambda\gamma\beta-1}} \left[ \begin{array}{l} \frac{H(1-\beta\lambda\gamma)-[1-(1-\alpha)\beta\lambda\gamma]kN_1}{H(1-\beta\lambda\gamma)-[1-(1-\alpha)\beta\lambda\gamma]kN_2} \\ \times \frac{H(1-\beta\gamma)-[1-(1-\alpha)\beta\gamma]kN_2}{H(1-\beta\gamma)-[1-(1-\alpha)\beta\gamma]kN_1} \end{array} \right]^{\frac{1}{\gamma}} \\ + N_2 \end{array} \right] \\ + M_2 \left[ N_2 + N_1 Z^{\frac{1}{\lambda\gamma\beta-1}} \right] \end{array} \right)} \right) \quad (\text{A.89})$$

Since  $Z < 1 \Rightarrow Z^{\frac{1}{\lambda\gamma\beta-1}} > 1$ , we have that  $\frac{pdf_{11}}{pdf_{12}} > 1$ . Since the distribution is symmetric, we also have that

$$\frac{pdf_{31}}{pdf_{32}} > 1 \quad \blacksquare$$

### III. Agglomeration Externalities

Since Marshall (1890), there is a broad consensus in the economics literature that the principal explanation for the existence of cities is the presence of agglomeration externalities (see Duranton and Puga, 2004 for a theoretical survey, and Rosenthal and Strange, 2004, and Combes, Duranton, Gobillon, Puga, and Roux, 2009 for the empirical evidence). Due to economies of scale and network effects, firms that cluster together may see a decline in the costs of production, due to the presence of competing multiple suppliers, greater specialization and the division of labor. Even direct competitors in the same sector may benefit because the cluster of firms in the same city attracts more suppliers and customers than a single firm in isolation. Alternatively, there may be demand-side agglomeration externalities due to the variety of goods and services provided. Of course, the size of the city is limited by diseconomies of agglomeration, for example congestion, and the limited availability of land which drives up the price of housing and office space. The latter is captured by the fixed amount of land in our baseline model above.

Because the key feature of agglomeration externalities is city size, we will assume that TFP is determined *endogenously* and increasing in city size  $S_j$ :  $A_j = A(S_j)$ , where  $A'(S) > 0$ . Cities are *ex ante* identical and there are no initial differences in TFP across different cities.

We analyze the case of Extreme-Skill Complementarities with free mobility of firms in a two-city model. The next Theorem characterizes the equilibrium allocation of skilled workers and of firms across cities. Firms in larger cities are more productive due to higher agglomeration externalities. Also real estate prices are higher. This may seem obvious, but as in the results on city size with exogenous TFP, this hinges on the fact that land supply is equal (or at least not too different). Once endogenous TFP due to agglomeration is higher, labor productivity is higher and therefore labor demand. As a result, the representative firm is larger and more firms enter. Finally, the same logic that explains the emergence of thick tails applies exactly as it does for the case of exogenous TFP.

**Theorem A. 3** *Given endogenous agglomeration externalities and given  $\lambda > 1$  and  $\lambda\gamma < 1$ , and provided cities of different size exist, the larger city has:*

- *higher TFP;*
- *higher real estate prices;*
- *more and larger firms;*
- *thicker tails in the skill distribution.*

**Proof.** See below. ■

Observe that with endogenous agglomeration externalities we can readily extend the proofs to the case of the technology with Top-Skill Complementarity.

An open question remains whether there actually exist equilibria with endogenous agglomeration externalities where cities are *ex post* heterogeneous, despite being *ex ante* identical. Of course, if there is heterogeneity in

equilibrium city size, we expect there to be multiple equilibria since there is no ex ante advantage to any city ex ante: one equilibrium where city 1 is large and city 2 is small, another equilibrium where city 2 is large and city 1 is small, and finally an intermediate equilibrium where cities are identical.

For the case of a CES production technology, we show conditions under which cities are different in size, despite the ex ante identical technologies and agglomeration externalities. Mobility of workers and free entry of firms induces wages and housing prices to adjust such that workers are indifferent between locating in either city. When there are sufficiently large economies of scale of agglomeration, i.e. when the function  $A$  is sufficiently convex, we obtain that cities differ in equilibrium. We establish this result for the exponential function in conjunction with the CES technology.

**Theorem A. 4** *Given the CES technology and endogenous agglomeration externalities of the form  $A(S) = e^{\psi S}$ ,  $\psi > 0$ , cities of different size exist, provided  $\psi > \frac{2(1-\gamma(1-\alpha))}{\bar{M}(1-\alpha)}$ .*

**Proof.** See below. ■

This result indicates that agglomeration externalities in production alone can generate the coexistence of cities of different size and productivity. The qualifier requires that for a given size of the labor force  $\bar{M}$ , the externality must be strong enough. If  $\psi$  is high enough, the function  $A = e^{\psi S}$  will be convex enough and as a result, there will be a large enough agglomeration effect that generates the existence of multiple equilibria.

Interestingly, a commonly assumed functional form in partial equilibrium,  $A(S) = S^\phi$ , does not generate heterogeneous cities in conjunction with the CES technology. This is true even if  $A$  is convex ( $\phi > 1$ ) as shown in the following Corollary. The reason is that already under CES, there is proportionality in the equilibrium demand for labor (proportional across skills), and an externality of this form lifts each city's productivity, but again proportionally. As a result homotheticity, the size of the city is fully governed by the decreasing returns at each skill level. The returns to scale can never be sufficiently strong.

**Corollary A. 2** *Given the CES technology and endogenous agglomeration externalities of the form  $A(S) = S^\phi$ ,  $\phi > 0$ , generically cities are of identical size.*

**Proof.** See below. ■

Ideally we would like to solve the model and prove that multiple equilibria exist also in the presence of extreme-skill complementarities. Unfortunately, that problem is quite a bit more challenging due to the dimensionality of the skill distribution. Not surprisingly, under CES, the proportionality of labor demand implies that distributions are identical across cities. As a result, we only need to solve for the endogenous city size, and not each skill level individually. While we cannot prove any general results, we do conjecture that the nature of the results extends to the non-CES case.

## Proofs Agglomeration Externalities

Going back to the system of five equations in the preliminaries, we can now substitute  $A_j$  for  $A(S_j)$ . Denote by  $\bar{M} = S_1 + S_2 = \sum_i M_i$  as the economy wide population. Then we will write  $S_2 = \bar{M} - S_1$ .

From dividing the first by the second and rearranging, we obtain:

$$\begin{aligned} & \left\{ \begin{array}{l} [1 - \lambda\gamma(1 - \alpha)] \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\bar{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} \\ - \frac{N_2}{N_1} \frac{p_1}{p_2} \frac{A(\bar{M} - S_1)}{A(S_1)} [1 - \lambda\gamma(1 - \alpha)] \end{array} \right\} \left( \frac{M_3}{N_2 + N_1 \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\bar{M} - S_1)}{A(S_1)} \right]^{\frac{1}{\lambda\gamma - 1}}} \right)^{\lambda\gamma} \left[ y_3 + \left( \frac{M_1}{M_3} \right)^\gamma y_1 \right]^\lambda \\ = & \left\{ \begin{array}{l} \frac{N_2}{N_1} \frac{p_1}{p_2} \frac{A(\bar{M} - S_1)}{A(S_1)} [1 - \gamma(1 - \alpha)] \\ - [1 - \gamma(1 - \alpha)] \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\bar{M} - S_1)}{A(S_1)} \right]^{\frac{\gamma}{\gamma - 1}} \end{array} \right\} \left( \frac{M_2}{N_2 + N_1 \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\bar{M} - S_1)}{A(S_1)} \right]^{\frac{1}{\gamma - 1}}} \right)^\gamma y_2, \end{aligned}$$

and from dividing the third by the fourth we have:

$$\begin{aligned} & \left\{ \begin{array}{l} (1 - \lambda\gamma) \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\bar{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} \\ - (1 - \lambda\gamma) \frac{p_1}{p_2} \frac{A(\bar{M} - S_1)}{A(S_1)} \end{array} \right\} \left( \frac{M_3}{N_2 + N_1 \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\bar{M} - S_1)}{A(S_1)} \right]^{\frac{1}{\lambda\gamma - 1}}} \right)^{\lambda\gamma} \left[ y_3 + \left( \frac{M_1}{M_3} \right)^\gamma y_1 \right]^\lambda \\ = & \left\{ \begin{array}{l} (1 - \gamma) \frac{p_1}{p_2} \frac{A(\bar{M} - S_1)}{A(S_1)} \\ - (1 - \gamma) \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\bar{M} - S_1)}{A(S_1)} \right]^{\frac{\gamma}{\gamma - 1}} \end{array} \right\} \left( \frac{M_2}{N_2 + N_1 \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\bar{M} - S_1)}{A(S_1)} \right]^{\frac{1}{\gamma - 1}}} \right)^\gamma y_2 \end{aligned}$$

Jointly, these two equations give us

$$\frac{[1 - \lambda\gamma(1 - \alpha)] \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\bar{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} - \frac{N_2}{N_1} \frac{p_1}{p_2} \frac{A(\bar{M} - S_1)}{A(S_1)} \right\}}{(1 - \lambda\gamma) \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\bar{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} - \frac{p_1}{p_2} \frac{A(\bar{M} - S_1)}{A(S_1)} \right\}} = \frac{[1 - \gamma(1 - \alpha)] \left\{ \frac{N_2}{N_1} \frac{p_1}{p_2} \frac{A(\bar{M} - S_1)}{A(S_1)} - \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\bar{M} - S_1)}{A(S_1)} \right]^{\frac{\gamma}{\gamma - 1}} \right\}}{(1 - \gamma) \left\{ \frac{p_1}{p_2} \frac{A(\bar{M} - S_1)}{A(S_1)} - \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\bar{M} - S_1)}{A(S_1)} \right]^{\frac{\gamma}{\gamma - 1}} \right\}} \quad (\star) \quad (\text{A.90})$$

We can now establish the following preliminary results. First, cities are not equal in the number of firms  $N_j$ .

**Lemma A. 2** *If  $\lambda > 1$  and  $\lambda\gamma < 1$ , then  $\frac{N_2}{N_1} \neq 1$ , and  $Z \neq 1$ .*

**Proof.** Assume  $\frac{N_2}{N_1} = 1$ , then from  $(\star)$  we have:

$$\frac{[1 - \lambda\gamma(1 - \alpha)]}{(1 - \lambda\gamma)} = \frac{[1 - \gamma(1 - \alpha)]}{(1 - \gamma)} \quad (\text{A.91})$$

which is a contradiction, since  $\lambda > 1 \Rightarrow \frac{[1 - \lambda\gamma(1 - \alpha)]}{(1 - \lambda\gamma)} > \frac{[1 - \gamma(1 - \alpha)]}{(1 - \gamma)}$ .

Rewriting the equality (★) above, we obtain:

$$\frac{[1 - \lambda\gamma(1 - \alpha)] \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} - \frac{N_2}{N_1} \frac{p_1}{p_2} \frac{A(\overline{M} - S_1)}{A(S_1)} \right\}}{(1 - \lambda\gamma) \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\gamma}{\gamma - 1}} - \frac{N_2}{N_1} \frac{p_1}{p_2} \frac{A(\overline{M} - S_1)}{A(S_1)} \right\}} = \frac{[1 - \gamma(1 - \alpha)] \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} - \frac{p_1}{p_2} \frac{A(\overline{M} - S_1)}{A(S_1)} \right\}}{(1 - \gamma) \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\gamma}{\gamma - 1}} - \frac{p_1}{p_2} \frac{A(\overline{M} - S_1)}{A(S_1)} \right\}} \quad (\text{A.92})$$

Then, if  $Z = 1$ , we have  $\left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} = \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\gamma}{\gamma - 1}} = 1$ . Therefore, we have:

$$\begin{aligned} \frac{[1 - \lambda\gamma(1 - \alpha)] \left\{ 1 - \frac{N_2}{N_1} \frac{p_1}{p_2} \frac{A(\overline{M} - S_1)}{A(S_1)} \right\}}{(1 - \lambda\gamma) \left\{ 1 - \frac{N_2}{N_1} \frac{p_1}{p_2} \frac{A(\overline{M} - S_1)}{A(S_1)} \right\}} &= \frac{[1 - \gamma(1 - \alpha)] \left\{ 1 - \frac{p_1}{p_2} \frac{A(\overline{M} - S_1)}{A(S_1)} \right\}}{(1 - \gamma) \left\{ 1 - \frac{p_1}{p_2} \frac{A(\overline{M} - S_1)}{A(S_1)} \right\}} \\ \frac{[1 - \lambda\gamma(1 - \alpha)]}{(1 - \lambda\gamma)} &= \frac{[1 - \gamma(1 - \alpha)]}{(1 - \gamma)} \end{aligned}$$

which as we saw before, it is a contradiction. ■

Given  $Z$ , we can now establish the main relations between the number of firms  $N_j$ , city size  $S_j$ , housing prices  $p_j$  and TFP  $A(S_j)$ .

**Lemma A. 3** *If  $Z < 1$ , then:*

1.  $N_1 > N_2$ ;
2.  $S_1 > S_2$ ;
3.  $A_1 > A_2$ ;
4.  $p_1 > p_2$ .

*With opposite inequalities if  $Z > 1$ .*

**Proof.** We establish each of the items in turn

1. Rearranging equality (★), we get:

$$\frac{[1 - \lambda\gamma(1 - \alpha)] \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} - \frac{N_2}{N_1} \frac{p_1}{p_2} \frac{A(\overline{M} - S_1)}{A(S_1)} \right\}}{(1 - \lambda\gamma) \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} - \frac{p_1}{p_2} \frac{A(\overline{M} - S_1)}{A(S_1)} \right\}} = \frac{[1 - \gamma(1 - \alpha)] \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\gamma}{\gamma - 1}} - \frac{N_2}{N_1} \frac{p_1}{p_2} \frac{A(\overline{M} - S_1)}{A(S_1)} \right\}}{(1 - \gamma) \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\gamma}{\gamma - 1}} - \frac{p_1}{p_2} \frac{A(\overline{M} - S_1)}{A(S_1)} \right\}} \quad (\text{A.93})$$

and further simplifying:

$$\frac{[1 - \lambda\gamma(1 - \alpha)] \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} \frac{p_2}{p_1} \frac{A(S_1)}{A(\overline{M} - S_1)} - \frac{N_2}{N_1} \right\}}{(1 - \lambda\gamma) \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} \frac{p_2}{p_1} \frac{A(S_1)}{A(\overline{M} - S_1)} - \frac{N_2}{N_1} \right\}} = \frac{[1 - \gamma(1 - \alpha)] \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} \frac{p_2}{p_1} \frac{A(S_1)}{A(\overline{M} - S_1)} - 1 \right\}}{(1 - \gamma) \left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} \frac{p_2}{p_1} \frac{A(S_1)}{A(\overline{M} - S_1)} - 1 \right\}} \quad (\text{A.94})$$

Then:

$$\frac{d}{d\frac{N_2}{N_1}}(LHS) = \frac{[1 - \lambda\gamma(1 - \alpha)]}{(1 - \lambda\gamma)} \frac{\left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\lambda\gamma}{\lambda\gamma - 1}} - \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\gamma}{\gamma - 1}} \right\} \frac{p_2}{p_1} \frac{A(S_1)}{A(\overline{M} - S_1)}}{\left\{ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(\overline{M} - S_1)}{A(S_1)} \right]^{\frac{\gamma}{\gamma - 1}} \frac{p_2}{p_1} \frac{A(S_1)}{A(\overline{M} - S_1)} - \frac{N_2}{N_1} \right\}^2} \quad (\text{A.95})$$

If  $Z < 1$ , we have:  $\frac{d}{d\frac{N_2}{N_1}}(LHS) > 0$ . Then, since  $\frac{[1 - \lambda\gamma(1 - \alpha)]}{(1 - \lambda\gamma)} > \frac{[1 - \gamma(1 - \alpha)]}{(1 - \gamma)}$ , if  $Z < 1$  we must have  $\frac{N_2}{N_1} < 1$ . Similarly, if  $Z > 1$ , we have:  $\frac{d}{d\frac{N_2}{N_1}}(LHS) < 0$ . Then, since  $\frac{[1 - \lambda\gamma(1 - \alpha)]}{(1 - \lambda\gamma)} > \frac{[1 - \gamma(1 - \alpha)]}{(1 - \gamma)}$ , if  $Z > 1$  we must have  $\frac{N_2}{N_1} > 1$ .

2. From the fifth equation, we have:

$$S_1 = (M_1 + M_3) \frac{Z^{\frac{1}{\lambda\gamma - 1}}}{\frac{N_2}{N_1} + Z^{\frac{1}{\lambda\gamma - 1}}} + \frac{Z^{\frac{1}{\gamma - 1}} M_2}{\frac{N_2}{N_1} + Z^{\frac{1}{\gamma - 1}}} \quad (\text{A.96})$$

If  $Z < 1$ , from the previous Lemma we know that  $\frac{N_2}{N_1} < 1$ . Since:

$$\begin{aligned} \frac{d}{d\lambda} \left( \frac{Z^{\frac{1}{\lambda\gamma - 1}}}{\frac{N_2}{N_1} + Z^{\frac{1}{\lambda\gamma - 1}}} \right) &= \frac{-\frac{\gamma}{(\lambda\gamma - 1)^2} Z^{\frac{1}{\lambda\gamma - 1}} \ln Z \times \left( \frac{N_2}{N_1} + Z^{\frac{1}{\lambda\gamma - 1}} \right) + \frac{\gamma}{(\lambda\gamma - 1)^2} Z^{\frac{1}{\lambda\gamma - 1}} Z^{\frac{1}{\lambda\gamma - 1}} \ln Z}{\left( \frac{N_2}{N_1} + Z^{\frac{1}{\lambda\gamma - 1}} \right)^2} \\ &= -\frac{\frac{\gamma}{(\lambda\gamma - 1)^2} \frac{N_2}{N_1} Z^{\frac{1}{\lambda\gamma - 1}} \ln Z}{\left( \frac{N_2}{N_1} + Z^{\frac{1}{\lambda\gamma - 1}} \right)^2} > 0 \text{ since } \ln Z < 0 \text{ as } Z < 1 \end{aligned}$$

We have:

$$S_1 > (M_1 + M_3 + M_2) \frac{Z^{\frac{1}{\gamma - 1}}}{\frac{N_2}{N_1} + Z^{\frac{1}{\gamma - 1}}} > \frac{\overline{M}}{2} \quad (\text{A.97})$$

The same logic establishes the opposite when  $Z > 1$ .

3. From the previous lemma, if  $Z < 1$ , we have that  $S_1 > S_2$ . Since  $A'(\cdot) > 0$ ,  $A(S_1) > A(\overline{M} - S_1) = A_2$ .

4. But then, from (★):

$$\frac{N_1}{N_2} \frac{A(S_1)}{A(\overline{M} - S_1)} \frac{\left\{ [1 - \lambda\gamma(1 - \alpha)] Z^{\frac{\lambda\gamma}{\lambda\gamma - 1}} \left( \frac{M_3}{N_2 + N_1 Z^{\frac{1}{\lambda\gamma - 1}}} \right)^{\lambda\gamma} \left[ y_3 + \left( \frac{M_1}{M_3} \right)^\gamma y_1 \right]^\lambda + [1 - \gamma(1 - \alpha)] Z^{\frac{\gamma}{\gamma - 1}} \left( \frac{Z^{\frac{1}{\gamma - 1}} M_2}{N_2 + N_1 Z^{\frac{1}{\gamma - 1}}} \right)^\gamma y_2 \right\}}{\left\{ [1 - \lambda\gamma(1 - \alpha)] \left( \frac{M_3}{N_2 + N_1 Z^{\frac{1}{\lambda\gamma - 1}}} \right)^{\lambda\gamma} \left( y_3 + \left( \frac{M_1}{M_3} \right)^\gamma y_1 \right)^\lambda + [1 - \gamma(1 - \alpha)] \left( \frac{M_2}{N_2 + N_1 Z^{\frac{1}{\gamma - 1}}} \right)^\gamma y_2 \right\}} = \frac{p_1}{p_2} \quad (\text{A.98})$$

Since  $Z < 1$ , we showed that  $\frac{N_1}{N_2} > 1$ . Then:

$$\frac{p_1}{p_2} > \frac{N_1}{N_2} \frac{A(S_1)}{A(\bar{M} - S_1)} Z^{\frac{\gamma}{\gamma-1}}. \quad (\text{A.99})$$

Similarly for  $Z > 1$ .

■

Next, we establish the result for thicker tails for the case of endogenous TFP from agglomeration externalities.

**Lemma A. 4** *Given  $\lambda > 1$  and  $\lambda\gamma < 1$ , the larger city has thicker tails.*

**Proof.** If  $Z < 1$ , from Lemma A.3 we know that city 1 is larger than city 2, and that  $A_1 > A_2$ . Therefore we can apply Theorem 2: city 1 is larger and has thicker tails. Instead, if  $Z > 1$ , we know that city 2 is larger than city 1, and that  $A_1 < A_2$ . Now we can define  $Z' = 1/Z$  (or relabel the cities) and again apply Theorem 2: city 2 is larger and has thicker tails. ■

Now our main result immediately follows from Lemmas A.2, A.3, and A.4:

**Theorem A.3** *Given endogenous agglomeration externalities and given  $\lambda > 1$  and  $\lambda\gamma < 1$ , and provided cities of different size exist, the larger city has:*

We now establish the proof of Theorem A.4

**Theorem A.4** *Given the CES technology and endogenous agglomeration externalities of the form  $A(S) = e^{\psi S}$ ,  $\psi > 0$ , cities of different size exist, provided  $\psi > \frac{2(1-\gamma(1-\alpha))}{M(1-\alpha)}$ .*

**Proof.** In the case of CES ( $\lambda = 1$ ) we can write the system of 9 equilibrium equations as:

$$\left\{ \begin{array}{l} m_{12} = \frac{M_1}{N_1 \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(S_2)}{A(S_1)} \right]^{\frac{1}{\gamma-1}} + N_2} \\ m_{22} = \frac{M_2}{N_1 \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(S_2)}{A(S_1)} \right]^{\frac{1}{\gamma-1}} + N_2} \\ m_{32} = \frac{M_3}{N_1 \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(S_2)}{A(S_1)} \right]^{\frac{1}{\gamma-1}} + N_2} \\ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(S_2)}{A(S_1)} \right]^{\frac{\gamma}{\gamma-1}} \{ (m_{12})^\gamma y_1 + (m_{32})^\gamma y_3 + (m_{22})^\gamma y_2 \} = \left[ \frac{H}{N_1} - k \right] \frac{p_1}{\alpha \gamma A(S_1)} \\ \{ (m_{12})^\gamma y_1 + (m_{32})^\gamma y_3 + (m_{22})^\gamma y_2 \} = \left[ \frac{H}{N_2} - k \right] \frac{p_2}{\alpha \gamma A(S_2)} \\ \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(S_2)}{A(S_1)} \right]^{\frac{\gamma}{\gamma-1}} \{ (m_{12})^\gamma y_1 + (m_{32})^\gamma y_3 + (m_{22})^\gamma y_2 \} = \frac{k}{(1-\gamma)A(S_1)} p_1 \\ \{ (m_{12})^\gamma y_1 + (m_{32})^\gamma y_3 + (m_{22})^\gamma y_2 \} = \frac{k}{(1-\gamma)A(S_2)} p_2 \\ S_1 = \left[ \left( \frac{p_1}{p_2} \right)^\alpha \frac{A(S_2)}{A(S_1)} \right]^{\frac{1}{\gamma-1}} [m_{12} + m_{22} + m_{32}] N_1 \\ S_2 = [m_{12} + m_{22} + m_{32}] N_2 \end{array} \right. \quad (\text{A.100})$$



From eqs. (6) and (7), we obtain:

$$\frac{\left[\left(\frac{p_1}{p_2}\right)^\alpha \frac{A(S_2)}{A(S_1)}\right]^{\frac{\gamma}{\gamma-1}} \{(m_{12})^\gamma y_1 + (m_{32})^\gamma y_3 + (m_{22})^\gamma y_2\}}{\{(m_{12})^\gamma y_1 + (m_{32})^\gamma y_3 + (m_{22})^\gamma y_2\}} = \frac{kp_1}{(1-\gamma)A(S_1)} \times \frac{(1-\gamma)A(S_2)}{kp_2}, \quad (\text{A.101})$$

and after rearranging:

$$\frac{p_1}{p_2} = \left(\frac{A(S_1)}{A(S_2)}\right)^{\frac{1}{1-\gamma(1-\alpha)}} \quad (\text{A.102})$$

From (4) and (6), and from (5) and (7) we have:

$$N_1 = \frac{H}{\left[1 + \frac{\alpha\gamma}{(1-\gamma)}\right]k} = N_2 \quad (\text{A.103})$$

Substituting (1), (2), and (3) into (8), and using the price ratio  $\frac{p_1}{p_2}$  we get:

$$S_1 = \left\{ \frac{\left(\frac{A(S_1)}{A(S_2)}\right)^{\frac{(1-\alpha)}{1-\gamma(1-\alpha)}}}{1 + \left(\frac{A(S_1)}{A(S_2)}\right)^{\frac{(1-\alpha)}{1-\gamma(1-\alpha)}}} \right\} \bar{M} \quad \text{and} \quad S_2 = \left\{ \frac{1}{1 + \left(\frac{A(S_1)}{A(S_2)}\right)^{\frac{(1-\alpha)}{1-\gamma(1-\alpha)}}} \right\} \bar{M} \quad (\text{A.104})$$

Then:

$$\frac{S_1}{S_2} = \left(\frac{A(S_1)}{A(S_2)}\right)^{\frac{(1-\alpha)}{1-\gamma(1-\alpha)}} \quad (\text{A.105})$$

Since  $S_2 = \bar{M} - S_1$ , we have:

$$\frac{S_1}{\bar{M} - S_1} = \left(\frac{A(S_1)}{A(\bar{M} - S_1)}\right)^{\frac{(1-\alpha)}{1-\gamma(1-\alpha)}} \quad (\text{A.106})$$

Now consider the case where  $A(S) = e^{\psi S}$  and denote the exponent on the RHS term by  $K = \frac{\psi(1-\alpha)}{1-\gamma(1-\alpha)}$  and observe that it is positive. Then the equilibrium condition is:

$$\begin{aligned} \frac{S_1}{\bar{M} - S_1} &= \left(e^{2S_1 - \bar{M}}\right)^K \\ \log\left(\frac{S_1}{\bar{M} - S_1}\right) &= K(2S_1 - \bar{M}) \end{aligned}$$

First, there is always a symmetric equilibrium  $S_1 = \frac{\bar{M}}{2}$ . Substituting  $S_1 = \frac{\bar{M}}{2}$  gives 0 both on the LHS and the RHS.

Next, we show that there are also two asymmetric equilibria, one where  $S_1 > \frac{\bar{M}}{2} > S_2$  and the mirror image with  $S_2 > \frac{\bar{M}}{2} > S_1$ . To see this, observe that the RHS is linear with bounded support on  $[0, \bar{M}]$  and takes values between  $-K\bar{M}$  and  $K\bar{M}$ . The LHS takes values between  $-\infty$  and  $+\infty$ : at  $S_1 = 0$ , the LHS is equal to

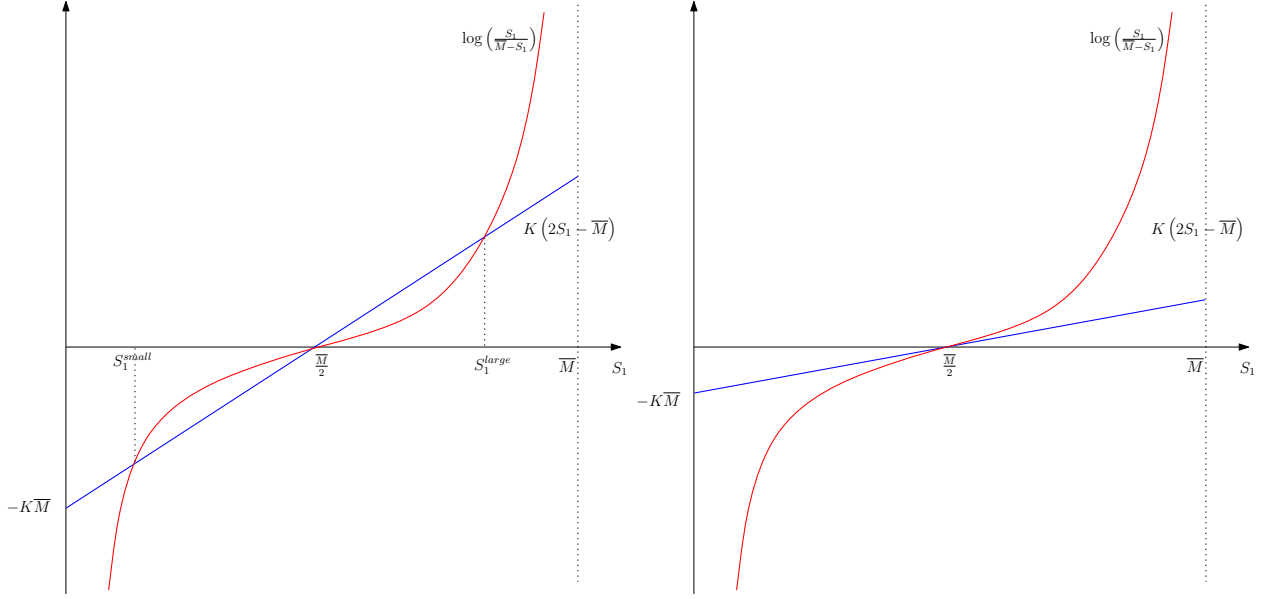


Figure 15: Proof of Theorem A.4: A. Multiple equilibria with cities of different sizes exist when  $K$  is large enough; B. a unique equilibrium exists with identical cities exists when  $K$  is small.

$\log 0 = -\infty$  and at  $S_1 = \bar{M}$ , the LHS is equal to  $\log \infty = +\infty$ . The slope of the LHS is positive and given by

$$\frac{\bar{M}}{S_1(\bar{M} - S_1)}. \quad (\text{A.107})$$

We know that there is an intersection at  $S_1 = \frac{\bar{M}}{2}$ , and therefore, given the behavior at  $S_1 = 0$  and  $\infty$  and continuity of both LHS and RHS, there are at least two more intersections provided the slope at  $S_1 = \frac{\bar{M}}{2}$  is flatter than the slope of the RHS, i.e. provided:

$$\frac{4}{\bar{M}} < 2K \quad \text{or} \quad \psi > \frac{2(1 - \gamma(1 - \alpha))}{\bar{M}(1 - \alpha)}. \quad (\text{A.108})$$

The logic is illustrated in Figure 15. ■

**Corollary A.2** *Given the CES technology and endogenous agglomeration externalities of the form  $A(S) = S^\phi$ ,  $\phi > 1$ , generically cities are of identical size.*

**Proof.** Now the equilibrium condition can be written as:

$$\frac{S_1}{\bar{M} - S_1} = \left( \frac{S_1}{\bar{M} - S_1} \right)^{\frac{\phi(1-\alpha)}{1-\gamma(1-\alpha)}}. \quad (\text{A.109})$$

which has a unique solution  $S_1 = \bar{M} - S_1$  provided  $\frac{\phi(1-\alpha)}{1-\gamma(1-\alpha)} - 1 \neq 0$ . When  $\frac{\phi(1-\alpha)}{1-\gamma(1-\alpha)} - 1 = 0$ , there is indeterminacy in the size of both cities and  $S_1 \in [0, \bar{M}]$ . However, this configuration of parameters is non-generic, therefore generically  $S_1 = \frac{\bar{M}}{2}$  and cities are identical. ■